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# On commuting flows of AKS hierarchy and twistor correspondence ★

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**Abstract**

The Adler–Kostant–Symes (AKS) scheme gives a geometrical method of construction of different integrable systems. In this paper we construct an AKS hierarchy, and we show that these commuting flows are reductions of self-dual Yang–Mills hierarchy.

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## 1. Introduction

It is well known that a systematic procedure of obtaining the most finite-dimensional completely integrable systems is the Adler, Kostant and Symes (AKS) theorem [AvM,Sy] to apply some Lie algebra $g$ equipped with an ad-invariant non-degenerate bilinear form. We assume that $g$, as a vector space $g = k + l$, is presented as the linear sum of two subalgebras. This bilinear form induces an isomorphism $g \simeq g^*$. Hence with the help of bilinear form $\langle\ ,\ \rangle$ we can identify $k^* \sim l^\perp$ and $l^* \sim k^\perp$ where

$$\langle k^\perp, k\rangle = \langle l^\perp, l\rangle = 0.$$

So $k^\perp$ acquires a Poisson structure from that of $l^*$. The coadjoint action of $L$ on $k^\perp \sim l^*$ is given by

$$g \circ p = \pi_{k^\perp}(gpg^{-1})$$

for $g \in L$ and $p \in k^\perp$. Then the infinitesimal action is $\eta(p) = \pi_{k^\perp}[\eta, p]$ for $\eta \in l$.

★ Dedicated to Prof. Amal Ray Chaudhuri on his 70th birthday with deep respect and admiration.


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The symplectic manifold here is some coadjoint $L$-orbit $M \subset k^{\perp} \simeq l^*$. We associate to it a Hamiltonian equation of suitable ad-invariant function $f : g \longrightarrow \mathbb{R}$ for all $f|_M$. In our case $g$ is a loop algebra. We mention that many important equations can be derived from this approach, e.g. Adler and van Moerbeke [AvM] obtained an Euler–Arnold equation and geodesic flow on ellipsoid, Ratiu [Ra] obtained C. Neumann equation and so on. In fact we have also obtained coupled KdV and non-linear Schrödinger equation by applying this AKS theorem. Hence AKS proves to be a general systematic procedure of obtaining many known and relatively unknown completely integrable Hamiltonian system.

In a recent paper Ablowitz et al. [ACT] have developed a hierarchy of self-dual Yang–Mills (SDYM) equations by introducing an operator. This approach to the SDYM hierarchy is based on the general concept of intertwining operators which was introduced by Schur. The concept of intertwining operator has been applied by Mikio Sato in his theory of KP system [SS]. These intertwiners transform "bare unperturbed" auxiliary linear systems into "perturbed" linear systems which have been treated as a formal power series in $\lambda^{-1}$. In fact Ablowitz et al. have shown that these intertwiners are the non-local functionals of the Yang–Mills potentials. As an application to two-dimensional systems, they have shown that upon appropriate reduction and suitable choice of gauge group this hierarchy produces all the well-known hierarchies of soliton equation in $1 + 1$-dimensions. Hence they call it universal integrable hierarchy.

Ward and Mason et al. [Wa,MS] have demonstrated that how the most well-known or not so well-known integrable systems arise as the reductions of SDYM equation. Thus all such integrable equations fall under the twistor framework. Certainly these give us a clue that we can reformulate the theory of integrable systems in terms of symmetry reductions of the SDYM equations. By reduction we mean: (a) one can reduce the number of independent variables to fewer than four by factoring out by a subgroup of the Poincaré group; and (b) one can reduce the number of dependent variables by imposing algebraic constraints on the connections.

This article serves two purposes: firstly, this gives a picture of higher flows for AKS hierarchy and we show that these are commuting flows, and secondly, since these flows are the reductions of SDYM hierarchy, morally this article gives further support to the twistor programme.

## 2. Adler–Kostant–Symes scheme

We separate out this section into two parts. In the first part we shall discuss the AKS theorem of a group with a two-cocycle [CG,Gu,RS]. In the second part we shall discuss the higher flows of AKS scheme, and we shall show that all the higher flows are commuting.

### *2.1. AKS theorem*

A two-form $\sigma$ on $M$ is called weak-symplectic form if $\omega$ is closed and the induced map from $TM$ to $T^*M$ is defined by

$$u \longmapsto \sigma(\circ, u),$$

if it is injective then $(M, \sigma)$ is called symplectic manifold. Let $M \subset k^{\perp} \simeq l^*$ be the coadjoint $L$ orbit. Then the weak-symplectic form is the Kostant–Kirillov two-form on $M$ defined by

$$\sigma_T(X, Y) = \langle T, [\xi, \eta] \rangle$$

for all $\xi, \eta \in g$ and $T \in M$.

Let $\Omega g = gl(n, C) \otimes C[\lambda, \lambda^{-1}]$ be the loop algebra of semi-infinite formal Laurent series in $\lambda$ with coefficients in $gl(n, C)$, e.g. an element $X(\lambda) \in \Omega g$ can be expressed as a formal series of the form

$$X(\lambda) = \sum_{i=\infty}^{m} x_i \lambda^i \quad \text{for all } x_i \in gl(n, C),$$

the Lie bracket with $Y(\lambda) = \sum_{j=-\infty}^{l} y_j \lambda^j$ is given by

$$[X(\lambda), Y(\lambda)] = \sum_{k=-\infty}^{m+1} \sum_{i+j=k} [x_i, y_j] \lambda^k.$$

We define a Poisson bracket of two smooth functions $g_1$ and $g_2$ on $\Omega g^*$ by

$$\{g_1, g_2\} = \langle \alpha, [\nabla g_1, \nabla g_2] \rangle,$$

where $\alpha \in \Omega g^*$, $\nabla$ is the usual gradient, i.e.

$$\langle \eta, \nabla g_i(\zeta) \rangle = \frac{\mathrm{d}}{\mathrm{d}t} g_i(\zeta + \eta t)|_{t=0} \quad \text{for all } \zeta, \eta \in g.$$

Here we will be working with the extended loop algebra $\tilde{\Omega} g$, the one-dimensional central extension of $\Omega g$, defined by the two-cocycle

$$\omega(X, Y) = \int_{S^1} X'Y \, \mathrm{d}x.$$

We define extended loop group $\Omega G$ to be

$$0 \longrightarrow \mathbb{R} \longrightarrow \tilde{\Omega} G \longrightarrow \Omega G \longrightarrow 1.$$

The corresponding loop algebra $\tilde{\Omega} g = \Omega g \oplus \mathbb{R}$. The Lie bracket of the loop algebra $\tilde{\Omega} g$ satisfies

$$[(X(\lambda), a), (Y(\lambda), b)] = \left( [X, Y], \int_{S^1} tr(XY') \right),$$

where $X \in \Omega g$ and $a \in \mathbb{R}$.

We define a non-degenerate ad-invariant bilinear form on $\Omega g$:

$$\langle X(\lambda), Y(\lambda)\rangle = \mathrm{res}_{\lambda=0}\, tr(\lambda^{-1}X(\lambda)Y(\lambda)) = tr(X(\lambda)Y(\lambda))_0$$

and this bilinear form can be extended to define the bilinear form on $\tilde{\Omega} g$ by

$$\langle (X, a), (Y, b)\rangle = ab + \int_{S^1} tr(XY).$$

There is a natural splitting in the loop algebra $\tilde{\Omega} g = \tilde{\Omega} g^+ \oplus \tilde{\Omega} g^-$, where $\tilde{\Omega} g^+$ denotes the subalgebra of $\tilde{\Omega} g$ given by the polynomial in $\lambda$, and $\tilde{\Omega} g^-$ is the subalgebra of strictly negative series.

Then via ad-invariant bilinear form $\langle\ ,\ \rangle$ we identify

$$(\tilde{\Omega} g^+)^* \sim (\tilde{\Omega} g^-)^\perp \text{ and } (\tilde{\Omega} g^-)^* \sim (\tilde{\Omega} g^+)^\perp.$$

Under the above identification, we can define the infinitesimal action of the coadjoint action of $\tilde{\Omega} G^-$ on $\tilde{\Omega} g^\perp$ and this is given by

$$(\pi_+(ad^*\, X)\mu + c\pi_+ X', 0).$$

The dual space $\tilde{\Omega} g^\perp$ stratifies into Poisson submanifolds corresponding to different values of the parameter; each of them is endowed with a Poisson bracket. Let us fix $c = 1$, so we confine us to a hyperplane in $\tilde{\Omega} g^*$. By abuse of notation we shall continue to call it $\tilde{\Omega} g^*$.

**Proposition 1.** *The Poisson bracket in the space of $\tilde{\Omega} g^*$ for the two smooth functions has the form*

$$\{f_1, f_2\}(Y) = \langle [\nabla f_1, \nabla f_2], Y\rangle + \int_{S^1} \nabla f_1 \frac{\mathrm{d}\nabla f_2}{\mathrm{d}x},$$

*where $Y \in \Omega g$.*

Let $I(g^*)$ denote the ring of infinitesimally $Ad^*$ invariant function on $g^* \oplus \mathbb{R}$. So $\nabla F \in I(g^*)$ will be ad-invariant function if and only if

$$\langle (\mu, 0), [X, \nabla F] + \partial \nabla F/\partial x\rangle = 0$$

for all $X \in g$ and $\mu \in g^*$ where $\nabla F$ is thought of as an element of $g \sim g^{**}$. In the absence of any central extension term $Ad^{\mathrm{ast}}$ invariant function satisfies

$$\langle \mu, [\nabla F, X]\rangle = 0.$$

Let $\hat{f}_1$ and $\hat{f}_2$ be the ad-invariant function and when they are restricted to $l^* \sim k^\perp$ these satisfy $\{\hat{f}_1, \hat{f}_2\}_{l^*} = 0$.

**Theorem 2.** *Let $\tilde{\Omega} g = \tilde{\Omega} g^+ \oplus \tilde{\Omega} g^-$ and $M \subset \tilde{\Omega} g^+$ a coadjoint orbit equipped with a natural weak orbit symplectic structure $\omega$. Let $H_i : \tilde{\Omega} g \longrightarrow \mathbb{R}$ be the set ad-invariant* *functions in $I(g^*)$ restricted to $(\tilde{\Omega} g^+)^\perp$ which is an involutive system on the coadjoint orbit. The Hamiltonian equations of motion on $\tilde{\Omega} g^*$ generated by the Hamiltonian (ad-invariant function) have the form*

$$\frac{\partial L}{\partial t} = \frac{\partial P}{\partial x} + [P, L],$$

*where $P = \pi_+[grad\, H]$.*

So this defines a flat connection $L\,dx + P\,dt$ on a cylinder $S^1 \times \mathbb{R}$ associated with the above zero curvature equation. In order to apply the AKS scheme we have to know about ad-invariant function. However Adler and van Moerbeke or Reiman and Semenov-Tian-Shansky gave a nice formalism to construct these functions. We will skip these discussions here.

Let us define the Hamiltonians by

$$H(\gamma) = \tfrac{1}{2} tr\, \lambda^{-1} \gamma^2,$$

where $\gamma$ is the orbit (which would be $L$ in the previous case). Let us assume our orbit to be $\gamma = \lambda^2 A + \lambda Q_2 + Q_1$, where $A$ is the constant diagonal matrix, and $Q_2$ and $Q_1$ are the off-diagonal and diagonal matrices, respectively. Although we take a very special orbit, this can be generalized to higher orbits.

Then the Hamiltonian equation would be

$$(\lambda^2 A + \lambda Q_1 + \lambda Q_2)_t = [\lambda^2 A + \lambda Q_1 + Q_2, \lambda A + Q_1] + (\lambda A + Q_1)_x.$$

**Remark.** When we apply Fordy–Kulish method to the AKS scheme we obtain different Hamiltonian systems associated to different hermitian symmetric spaces. In this case $Q_1$ and $Q_2$ will take special values which depend on the breaking of the Lie algebra (for details see [Ma]).

We shall now investigate the higher flows of the AKS scheme generated by the traces of higher powers of $A$.

## 3. AKS hierarchy and commuting flows

Let $\hat{g}((\lambda^{-1}))$ be an element of the loop algebra, and $\hat{g}|_+ = \sum_{0\leq k}^{\ll\infty} \oplus\, \hat{g}\lambda^k$ be its polynomial part and $\hat{g}|_- = \sum_{k\leq -1} \oplus\, \hat{g}\lambda^k$ its pure Laurent part. We assume that $\tilde{G}|_-$ be the group corresponding to subalgebra $\hat{g}|_-$.

Let $\gamma$ be the orbit defined as $\gamma := Q_1 + \lambda Q_2 + \lambda^2 A$. Hence the gradient of the Hamiltonian would be $\nabla H = Q_2 + \lambda A$. We will denote $\nabla H$ by $\gamma_2$.

**Lemma 3.** *There exists $S(x, \lambda) \in \hat{G}|_-$ such that it satisfies*

$$\partial_x - Q_1 - \lambda Q_2 - \lambda^2 A = S(\partial_x - \lambda^2 A) S^{-1}.$$

*Proof.* We shall follow the proof of Ablowitz et al. Here $S$ is a Sato type operator introduced by them in this context. It is easy to see that the above expression reduces to

$$\partial S S^{-1} = \Gamma - \lambda^2 S A S^{-1}.$$

Assume

$$\partial S S^{-1} = \sum_{n=0}^{\infty} \frac{V_n}{\lambda^n} \quad \text{and} \quad S A S^{-1} = A + \sum_{n=1}^{\infty} \frac{S_n}{\lambda^n}.$$

Here $V_n, S_n \in \hat{g}$, substituting this expression to the equation we obtain

$$S_2 = Q_1 - V_0, \quad S_1 = Q_2 \quad \text{and} \quad S_n = -V_{n-2}, \quad n > 2.$$

Consider the differential equation

$$\partial(S A S^{-1}) = [\partial S S^{-1}, S A S^{-1}] = [\gamma, S A S^{-1}],$$

this produces a recursion relation among $S_n$'s:

$$(ad\ A) S_{n+2} + (ad\ Q_2) S_{n+1} = (\partial - ad\ Q_1) S_n.$$

Since the coefficients $S_n$ and $V_n$ can be determined recursively, the existence of $S(x, \lambda) \in \hat{G}|_-$. □

**Definition 4.**
(1) We define

$$\phi := S A S^{-1} = \sum_{0}^{\infty} \frac{\phi_n}{\lambda^n},$$

where $\phi_0 = A$ and $\phi_1 = Q_2$.
(2) $\partial_k := \partial_{tk}$ and $\gamma_k = (\lambda^{k-1}\phi)_+$ for all $n \geq 0$, where $\gamma_2 := Q_2 + \lambda A$.

**Lemma 5.** *Coefficients $\phi_n \in \hat{g}, n \geq 1$, are uniquely determined by the initial conditions $\phi_0 = A$ and $\phi_1 = Q_2$.*

*Proof.* We know that $\phi = S A S^{-1}$ and $\partial_x - \gamma = S(\partial_x - \lambda^2 A) S^{-1}$, hence we obtain

$$[\partial - \gamma, \phi] = S[\partial_x - \lambda^2 A, A] S^{-1} = 0.$$

Thus we get $\partial_x \phi = [\gamma, \phi]$. Now we consider the expansion $\phi = \sum_{n=0}^{\infty} (\phi_n / \lambda^n)$ and $\gamma = \lambda^2 A + \lambda Q_2 + Q_1$. So using this expression we obtain the recursion relations

$$(\partial_x - ad\ Q_1)\phi_n + (ad\ A)\phi_{n+2} - (ad\ Q_2)\phi_{n+1} = 0.$$

Hence once we specify the initial conditions $\phi_0 = A$ and $\phi_1 = Q_2$ we can uniquely obtain all other $\phi_n$ recursively. □

**Definition 6.** An AKS hierachy is defined by the infinite sequence of flows with respect to higher times $t_k$ and the $k$th member of the family is given by

$$[\partial_x - \gamma, \partial_k - \gamma_k] = 0, \qquad k \geq 2,$$

i.e.

$$\partial_k \gamma - \partial_x \gamma_k + [\gamma, \gamma_k] = 0.$$

This zero curvature equation arises as the compatibility condition for the following linear systems:

$$\partial_x \xi = \gamma \xi \quad \text{and} \quad \partial_k \xi = \gamma_k \xi.$$

**Definition 7.** The $k$th flow in the AKS hierarchy can be considered as a "dressing" of a "bare" solution by the intertwiner

$$\partial_x - \gamma = S(\partial - \lambda^2 A)S^{-1}, \qquad \partial_k - \gamma_k = S(\partial_k - \lambda^{k-1} A)S^{-1}.$$

Then by simple rearrangement and manipulation we obtain

$$\gamma_k = \partial_k S S^{-1} + \lambda^{k-1} S A S^{-1} = \partial_k S S^{-1} + \lambda^{k-1} \phi.$$

Since by definition we know that $\gamma_k = (\lambda^{k-1}\phi)_+$ and $\partial_k S S^{-1} \in \hat{g}|_-$, we obtain

$$\partial_k S S^{-1} = -(\lambda^{k-1}\phi)_-. \tag{$*$}$$

**Lemma 8.** *The evolution equation for $\phi$ is given by $\partial_k \phi = [\gamma_k, \phi]$.*

*Proof.* We use the following equalities:

$$\partial_k - \gamma_k = S(\partial_k - \lambda^{k-1} A)S^{-1} \quad \text{and} \quad \phi = S A S^{-1}.$$

Hence it is not hard to see

$$[\partial_k - \gamma_k, \phi] = S[\partial_k - \lambda^{k-1} A, A]S^{-1} = 0.$$ □

**Theorem 9.** *The higher flows of the AKS are commuting.*

*Proof.* Let us rewrite Eq. (∗) by

$$\partial_k S = -(\lambda^{k-1}\phi)_- S \quad \text{for } k \geq 2.$$

The compatibility condition gives us

$$\partial_k (\lambda^{l-1}\phi)_- - \partial_l (\lambda^{k-1}\phi)_- + [(\lambda^{k-1}\phi)_-, (\lambda^{l-1}\phi)] = 0, \tag{E$_1$}$$

and this is valid only for $k, l \geq 2$. Now we shall use Lemma 8, we multiply $\lambda^{l-1}$ to that equation for $k$th time and $\lambda^{k-1}$ to that of $l$th time and then subtracting these two, we obtain the following equation:

$$\partial_l (\lambda^{k-1}\phi) - \partial_k (\lambda^{l-1}\phi) = [\gamma_l, (\lambda^{k-1}\phi)] - [\gamma_k, (\lambda^{l-1}\phi)].$$

When we project this equation to "negative" loops or $\hat{g}|_-$ we get

$$\partial_l(\lambda^{k-1}\phi)_- - \partial_k(\lambda^{k-1}\phi)_- = [\gamma_l, (\lambda^{k-1}\phi)]_- [\gamma_k, (\lambda^{l-1}\phi)]_-. \tag{E$_2$}$$

Hence using ($E_1$) and ($E_2$) we obtain

$$\begin{aligned} &-[(\lambda^{l-1}\phi)_+, (\lambda^{k-1}\phi)]_- + [(\lambda^{k-1}\phi)_+, (\lambda^{l-1}\phi)]_- + [(\lambda^{k-1}\phi)_-, (\lambda^{l-1}\phi)_-]_- \\ &\quad = [(\lambda^{k-1}\phi), (\lambda^{l-1}\phi)]_- = 0, \end{aligned}$$

where we have used $\gamma_k := (\lambda^{k-1}\phi)_+$. □

## 4. Reduced SDYM hierarchy and twistor correspondence

In this section we shall establish a link between the SDYM equation (hierarchy) and the AKS scheme (hierarchy). We shall begin this part with some preliminary definitions.

Let $V$ be a trivial vector bundle over $\mathbb{R}^4$ with fibres isomorphic as linear vector spaces to the Lie algebra $g$. We define that the connection and the curvature are $g$-valued one- and two-forms, respectively, and these are given by $A := \sum_{\mu=0}^{3} A_\mu(x)\,\mathrm{d}x^\mu$ and $F := \sum_{\mu\nu=0}^{3} F_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, where $x = (x^\mu)$ are the usual coordinates of $\mathbb{R}^4$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu].$$

**Definition 10.**

(1) The SDYM equations are the system of first-order partial differential equations given by

$$F_{01} = F_{23}, \qquad F_{02} = F_{31}, \qquad F_{03} = F_{12},$$

where connections are defined upto gauge transformation.

(2) In terms of complex coordinates on $\mathbb{R}^4$ the SDYM equations can be rewritten as

$$F_{yz} = F_{\bar{y}\bar{z}} = 0, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0,$$

where we introduce $y = x^1 + \mathrm{i}x^2$, $z = x^0 - \mathrm{i}x^3$, $\bar{y}$ and $\bar{z}$ are the complex conjugates.

(3) The SDYM equations can be realized as the compatibility condition for the following linear system:

$$D_1\psi = A_1\psi, \qquad D_2\psi = A_2\psi,$$

where

$$\begin{aligned} D_1 &= \partial_y + \lambda\partial_{\bar{z}}, & D_2 &= \partial_z - \lambda\partial_{\bar{y}}, \\ A_1 &= A_y + \lambda A_{\bar{z}}, & A_2 &= A_z - \lambda A_{\bar{y}}, \end{aligned}$$

and $\lambda \in CP^1$ is the spectral parameter.

These linear pairs of the SDYM were introduced by Belavin and Zakharov [BZ]. Under a suitable gauge transformation Ablowitz et al. have shown that the SDYM equations can be reduced to

$$F_{yz} = 0, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0.$$

In this representation $A_{\bar{y}}$ and $A_{\bar{z}}$ are the diagonal matrices.

The SDYM equations can be dimensionally reduced and when we reduce to $1+1$-dimensions, these equations depend only on the coordinates $y$ and $z$. Here we shall denote $y$ and $z$ by $x$ and $t$, respectively. We choose $A_{\bar{z}} = A^1$, $A_{\bar{y}} = -A^2$ and $A_y = U_1$, $A_z = U_2$ such that $A^i$s and $U^i$s are diagonal and off-diagonal matrices, respectively.

**Definition 11.** The $1+1$-dimensional reduction of the SDYM can be obtained from the compatibility condition of

$$\partial_x \psi = (U^1 + \lambda A^1)\psi, \qquad \partial_t \psi = (U^2 + \lambda A^2)\psi,$$

i.e. these look like

$$\partial_t U^1 - \partial_X U^2 + [U^1, U^2] = 0, \qquad [A^1, U^2] = [A^2, U^1].$$

**Theorem 12** [ACT].

(1) *The reduced SDYM hierarchy satisfies the following evolution equation at k-th time:*

$$\partial_k U^1 = \partial_x \chi_{k-1} - [U^1, \chi_{k-1}], \qquad t_2 = t, \quad k \geq 2,$$

*where $\chi$ is defined by*

$$\chi = A^2 + \sum_{k=1}^{\infty} \frac{\chi_k}{\lambda^k}.$$

(2) *These hierarchies can be realized as the compatibility condition of following linear system:*

$$\partial_k \psi = W_k \psi, \qquad k \geq 1,$$

*where $t_1 = x$, $W_1 = U^1 + \lambda A^1$ and $W_k = (\lambda^{k-1}\chi)_+$.*

(3) *Higher flows commute:*

$$\partial_k W_l - \partial_l W_k + [W_l, W_k] = 0.$$

Next we shall show that our AKS equation is a special case of $1+1$-dimensional reduced SDYM equation.

**Lemma 13.** *AKS system is $1+1$-reduction of the SDYM equation.*

*Proof.* It is easy to see that when we replece $A^1 = \lambda A$, $U^1 = Q_1 + \lambda Q_2$, $A^2 = A$ and $U^2 = Q_2$ we obtain the AKS equation. Moreover, if we put these values into the constraint equation $[A^1, U^2] = [A^2, U^1]$, we obtain $[A, Q_1]$. Since $A$ is a constant diagonal matrix and $Q_1$ is also a diagonal matrix so they commute, i.e. $[A, Q_1] = 0$. □

By a similar calculation the following result can be proved easily:

**Theorem 14.** *The AKS hierarchy is the reduction of the SDYM hierarchy.*

Since the AKS equations are reductions of the SDYM equations on $\mathbb{R}^4$, the standard twistor correspondence for the full SDYM equation can be reduced to give a correspondence for solutions of the AKS scheme.

Let us recall that the Bogomolnýi equation is obtained when single non-null translational symmetry is imposed on the SDYM equation. The solutions of the Bogomolnýi equation correspond to bundles invariant under the corresponding symmetry on $CP^3$ (twistor space). Since there is no fixed point on $CP^3$, here bundles are the pull back bundles on the quotient of $CP^3$ by that symmetry. This quotient is called minitwistor space and it is denoted by $\mathcal{O}(2)$, it is a holomorphic line bundle of Chern class 2 over $CP^1$. This idea can be generalized to any member of the Bogomolnýi hierarchy.

**Theorem 15** [MS]. *There exist a* 1:1 *correspondence between solution of the* $n$*-th Bogomolnýi hierarchy on a domain* $U \subset C^{n+1}$ *and the holomorphic vector bundle on the open region in* $\mathcal{O}(n)$ *swept out by the sections of* $\mathcal{O}(n)$ *corresponding to the points of* $U$. *This bundle is trivial when it is restricted to the sections of* $\mathcal{O}(n)$ *corresponding to points of* $U$.

In the case of the AKS scheme we impose one more additional symmetry. It is not possible to factor out this extra symmetry. Hence the solutions of the AKS equation have one to one correspondence with the holomorphic vector bundles satisfying certain symmetry and reality condition on $\mathcal{O}(2)$.

The solutions of the first $(n-1)$ equations of the hierarchy correspond to holomorphic vector bundles on $\mathcal{O}(n)$ satisfying the appropriate symmetry and reality condition.

## 5. Summary

In this paper we have given a hierarchy of the AKS scheme and shown that these are the reductions of the SDYM hierarchy proposed by Ablowitz et al. We know that the AKS scheme is a general method to derive integrable systems, hence this paper has strengthened the aim to reformulate the theory of integrable systems in terms of symmetry reductions of the SDYM equation and their twistor correspondences.

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## References

[ACT] M.J. Ablowitz, S. Chakravarty and L.J. Takhtajan, Comm. Math. Phys. 158 (1993) 289–314.
[AvM] M. Adler and P. van Moerbeke, Adv. Math. 38 (1980) 267.
[At] M.F. Atiyah, *Classical Geometry of Yang–Mills Fields*, Fermi Lectures, Scuola Normale Pisa (1980).
[BZ] A.A. Belavin and V.E. Zakharov, Phys. Lett. B 73 (1978) 53.
[CG] A.R. Chowdhury and P. Guha, J. Phys. A 23 (1990) L639.
[Gu] P. Guha, Loop groups, hermitian symmetric spaces and integrable systems, preprint (1994).
[Ma] I. Marshall, Phys. Lett. A 127 (1988) 19.
[MS] L.J. Mason and G. Sparling, Phys. Lett. A 137 (1989) 29–33.
[Ra] T. Ratiu, Trans. Amer. Math. Soc. 264(2) (1981) 321–329.
[RS] A.G. Reiman and M.A. Semenov-Tjan-Sanskii, Soviet Math. Dokl. 21 (1985) 2.
[SS] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, Lect. Notes in Numer. Appl. Anal. 5 (1982) 259–271.
[Sy] W. Symes, Invent. Math. 159 (1980) 13.
[Wa] R.S. Ward, Phys. Lett. A 61 (1977) 81.